Space-Time Residual Distribution Schemes for Hyperbolic Conservation Laws over Linear and Bilinear Elements

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Abstract

The aim of this lecture note is to summarize the latest developments on a special class of space-time methods designed to obtain the time accurate solution of hyperbolic scalar and systems of conservation laws. In the present class of schemes the spatial and the temporal derivatives are not distinguished in any way. The application of existing multidimensional upwind residual distribution schemes over properly constructed space-time meshes naturally leads to an unconditionally stable implicit time stepping procedure, maintaining both the higher-order accuracy of the solution in smooth unsteady flows and the monotone resolution of discontinuities. Both linear and bilinear space-time elements are considered. The resulting schemes are extensively tested on several scalar test problems and on the solution of the Euler and the ideal magnetohydrodynamic equations. The method has a great potential for applications involving moving grids with dynamic mesh adaptation.

1 Preface

This lecture note is devoted to the extension of the residual distribution ($\mathcal{RD}$) method to the context of space-time elements in order to achieve the time accurate solution of hyperbolic conservation laws while maintaining the accuracy and the monotone shock capturing properties of the underlying basic schemes. For the sake of easy understanding the presented material is divided into five logically separated parts. In part I the basic philosophy behind the construction of our space-time method is introduced and the corresponding numerical schemes are readily derived for the solution of hyperbolic scalar conservation laws in one and two spatial dimensions over linear space-time elements. In part II the method is extended to non commuting hyperbolic systems of equations. A comprehensive evaluation of the schemes is presented for the one and two dimensional cases in parts III and IV, respectively. In part V the basic priciples presented in part I are extended to prismatic space-time elements involving a bilinear approximation of the solution for the two dimensional scalar case. Note, that the accompanying lecture note of Mezine *et. al* [25] offers an alernative to our approach within the context of space-time residual distribution.

It is assumed that the reader is already familiar with the fundamentals of the residual distribution method applied to the steady solution of hyperbolic conservation laws, as reported *e.g.* in [26, 34], and recalled in the accompanying lecture notes of Prof. Deconinck [15] and Prof. Abgrall [3]. The reader may also wish to consult the related references [9, 10, 11, 12, 13] for the preliminaries in the given framework, omitted here. To avoid possible confusion due to different labeling conventions, and to keep this material easy to read, in the rest of this section we briefly recall the basic definitions employed through this note.

Labeling Conventions and Definitions: Scalar Case

Consider the solution of a hyperbolic scalar conservation law in d spatial dimensions defined over domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, written in its conservative and quasilinear forms as:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F} = 0, \qquad \frac{\partial u}{\partial t} + \boldsymbol{\lambda} \cdot \nabla u = 0, \tag{1}$$

respectively, where u is the conserved quantity, $\mathbf{F} = (F_1, ..., F_d)$ is the corresponding flux function and the advection speed vector is

$$\boldsymbol{\lambda} = \sum_{m=1}^{d} \left(\hat{\mathbf{x}}_{\mathbf{m}} \frac{\partial F_m}{\partial u} \right). \tag{2}$$

As a convention, the unit vector in the m-th coordinate direction and the spatial position vector itself are labeled by $\hat{\mathbf{x}}_{\mathbf{m}}$ and $\mathbf{x}$, respectively. In the framework of the $\mathcal{RD}$ method the analytic solution u of equation (1) is approximated by u^h in the linear finite element space over an arbitrary unstructured grid mapping Ω:

$$u^h(\mathbf{x},t) = \sum_{j=1}^{\mathcal{G}} u_j(t)\,\omega_j^h(\mathbf{x}), \tag{3}$$

where $\mathcal{G}$ is the total number of nodes in the mesh, $u_j(t)$ is the time dependent value of the solution at node j, and $\omega_j^h(\mathbf{x})$ is the piecewise linear shape function equal to unity at node j and vanishing outside of the elements sharing j as a vertex. In element E the cell residual ϕ^E is defined by

$$\phi^E = \oint_{\partial E} \mathbf{F}(u^h)\cdot\mathbf{n}\,dS \equiv \int_E \boldsymbol{\lambda}\cdot\nabla u^h\,dV, \tag{4}$$

where $\mathbf{n}$ is the outward pointing unit normal of surface element dS. The inward pointing normal vector of the face opposite to node i scaled with the area of the face is denoted by $\mathbf{n_i}$. The distribution function describing the portion of the cell residual distributed to node i in element E is labeled by ϕ_i^E. The multidimensional upwind distribution of ϕ^E is incorporated into the schemes via upwind parameter k_i defined as

$$k_i = \frac{\bar{\boldsymbol{\lambda}}\cdot\mathbf{n_i}}{d}. \tag{5}$$

Based on the orientation of the elementwise constant advection speed vector $\bar{\boldsymbol{\lambda}}$, we distinguish *upstream* and *downstream* nodes in element E. For the sake of simple terminology the following convention is adopted:

Definition 1.1.

$$\begin{array}{llllll} \textbf{Downstream node:} & k_i^+ = k_i & \text{and} & k_i^- = 0 & \text{if} & k_i > 0, \\ \textbf{Upstream node:} & k_i^+ = 0 & \text{and} & k_i^- = k_i & \text{if} & k_i \le 0. \end{array} \tag{6}$$

Assuming $|\bar{\boldsymbol{\lambda}}| \neq 0$, there is at least one upstream and one downstream node in each element. The multidimensional upwind property of the $\mathcal{RD}$ schemes implies that no residual is sent to upwind nodes, it is distributed between the downstream vertices, *i.e.*:

Definition 1.2. A scalar $\mathcal{RD}$ scheme is *multidimensional upwind* ($\mathcal{MU}$) if

$$\phi_i^E = 0 \qquad \text{for} \qquad k_i^+ = 0. \tag{7}$$

Labeling Conventions and Definitions: System Case

Consider the solution of a hyperbolic system of q conservation laws written in its conservative and quasilinear forms as:

$$\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{F} = 0, \qquad \frac{\partial U}{\partial t} + \sum_{m=1}^{d} A_m^U \frac{\partial U}{\partial x_m} = 0, \tag{8}$$

respectively, where U is the state vector of q conserved quantities, $\mathbf{F} = (F_1, ..., F_d)$ is the corresponding flux function. The m-th Jacobian of the system with respect to U and to a vector of auxilary unknowns W are, respectively:

$$A_m^U = \frac{\partial F_m}{\partial U}, \qquad \text{and} \qquad A_m^W = \frac{\partial F_m}{\partial W}. \tag{9}$$

The analytic solution W is approximated by W^h in the piecewise linear finite element space over an arbitrary unstructured grid mapping Ω:

$$W^h(\mathbf{x},t) = \sum_{j=1}^{\mathcal{G}} W_j(t)\,\omega_j^h(\mathbf{x}), \tag{10}$$