Analysis of regularization

- Let us denote by $\delta(\Gamma, g, \mathbf{x})$ a delta function of variable strength supported on Γ, such that

$$\int_\Omega \delta(\Gamma, g, \mathbf{x}) d\mathbf{x} = \int_\Gamma g(S) dS.$$

- With this notation, the surface tension forces in the Navier-Stokes equations are given by

$$\mathbf{f}(\mathbf{x}, t) = \sigma \delta(\Gamma, \kappa \hat{\mathbf{n}}, \mathbf{x}).$$

- Very complex to analyze full problem.
- Study regularization of a source term of the same kind in a linear PDE.
- First assume a uniform grid, and later a general irregular grid.

Use of Greens functions

Let the solution of a differential equation

$$\begin{aligned} Lu &= s(\mathbf{x}) \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \\ Bu &= r(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{1}$$

be given on the standard form as an integral of the fundamental solution $G(\mathbf{x}, \mathbf{y})$ multiplying the source term $s(\mathbf{x})$,

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) s(\mathbf{y}) d\mathbf{y} + R(\mathbf{x}), \tag{2}$$

where $R(\mathbf{x})$ represents the contribution from the boundary conditions.

For a uniform grid: The solution of the corresponding numerical approximation to (1) at $\mathbf{x_j}$, can be written as

$$u_{\mathbf{j}} = \left(\prod_{k=1}^{d} h_k \right) \sum_{\mathbf{m} \in \Omega_h} G_{\mathbf{jm}} s_{\mathbf{m}} + R_{\mathbf{j}}, \tag{3}$$

where $G_{\mathbf{jm}}$ is the discrete fundamental solution and Ω_h is the index set for the grid points inside Ω. $R_{\mathbf{j}}$ is the contribution from the BC.