Unstructured Grid Generation by Delaunay Triangulation

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1. Geometrical Construction

Dirichlet[1] in 1850 proposed a method whereby a given domain could be systematically decomposed into a set of packed convex polygons. Given two points in the plane, p_i and p_j , the perpendicular bisector of the line joining the two points subdivides the plane into two regions, V_i and V_j . The region V_i is the space closer to p_i than to p_j . Extending these ideas, it is clear that for a given set of points in the plane, the regions V_i are territories which can be assigned to each point such that V_i represents the space closer to p_i than to any other point in the set. This geometrical construction of tiles is known as the Dirichlet tessellation. This tessellation of a closed domain results in a set of non-overlapping convex polygons, called Voronoi regions, covering the entire domain [2,3].

A more formal definition can be stated. If a set of points is denoted by $\{p_i\}$, then the Voronoi region $\{V_i\}$ can be defined as

$$\{V_i\} = \{ p : \| p-p_i \| < \|p-p_j \|, \ \forall j \neq i \}$$

i.e. the Voronoi region $\{V_i\}$ is the set of all points that are closer to p_i than to any other point. The sum of all points forms a Voronoi polygon.

From this definition, it is apparent, that in two dimensions, the territorial boundary which forms a side of a Voronoi polygon must be midway between the two points which it separates and is thus a segment of the perpendicular bisector of the line joining these two points. If all point pairs which have some segment of boundary in common are joined by straight lines, the result is a triangulation of the convex hull of the set of points $\{p_i\}$. This triangulation is known as the Delaunay triangulation. An example of this construction, illustrated in two dimensions is shown in Figure 1.1.

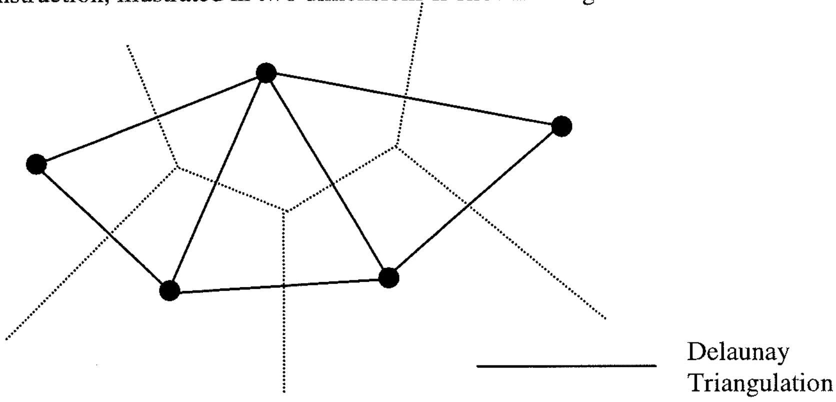

Figure 1.1 The Delaunay and Voronoi constructions.

Equivalent constructions can be defined in higher dimensions. In three dimensions, the territorial boundary which forms a face of a Voronoi polyhedron is equidistant between the two points which it separates. If all point pairs which have a common face in the Voronoi construction are connected then a set of tetrahedra is formed which covers the convex hull of the data points.

The Delaunay triangulation has some rather interesting properties. One of particular interest is the so-called in-circle criterion. The vertices of the Voronoi diagram are at the circumcentres of the circles which pass through the three points which form each triangle. In three dimensions, the Voronoi vertices are at the circumcentre of the spheres which pass through the four points which form each tetrahedron. It follows from the definition of the Dirichlet tessellation that no points, other than the so-called forming points which form the triangles or tetrahedra, fall within the circles or spheres. If a point did fall inside then this would contradict the basic definition. This geometrical property is the in-circle criterion (Figure 1.2).

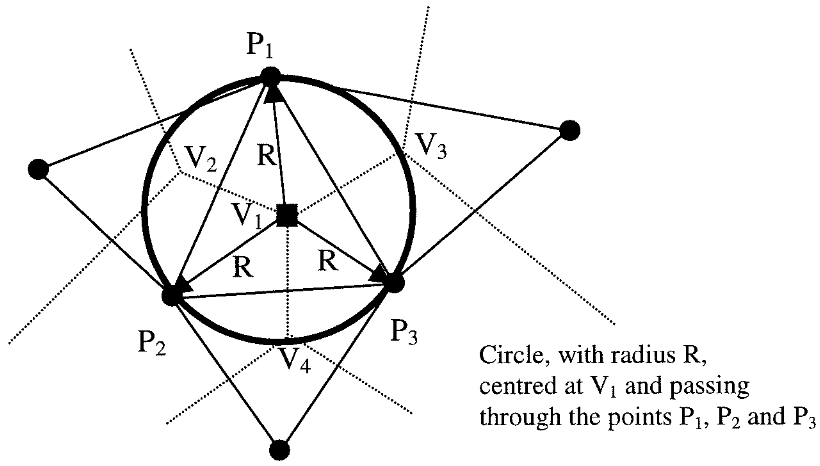

Figure 1.2 The Delaunay and Voronoi constructions.

2. An algorithm to construct the Delaunay triangulation

The in-circle criterion forms the basis for the most popular algorithm for the construction of the tessellation which was proposed by Bowyer and Watson [4,5]. The algorithm is applicable in both 2 and 3 dimensions.

The basic outline of the algorithm, presented here for 3 dimensions is;-

1. Define the convex hull within which all points will lie. It is appropriate to specify 8 points together with the associated Voronoi diagram structure.
2. Introduce a new point anywhere within the convex hull {1,..,8}.
3. Determine all vertices of the Voronoi diagram to be deleted. A point which lies within the sphere, centred at a vertex of the Voronoi diagram and which passes through its four forming points, results in the deletion of that vertex. This follows from the 'in-circle' definition of the Voronoi construction.
4. Find the forming points of all the deleted Voronoi vertices. These are the contiguous points to the new point.
5. Determine the Voronoi vertices which have not themselves been deleted which are neighbours to the deleted vertices. These data provide the necessary information to enable valid combinations of the contiguous points to be constructed.
6. Determine the forming points of the new Voronoi vertices. The forming points of new vertices must include the new point together with the three points which are contiguous to the new point and form a face of a neighbouring tetrahedra (these are the possible combinations obtained from Step 5).
7. Determine the neighbouring Voronoi vertices to the new Voronoi vertices. Following Step 6, the forming points of all new vertices have been computed. For each new vertex, perform a search through the forming points of the neighbouring vertices as found in Step 5 to identify common pairs of forming points. When a common combination occurs, then the three associated vertices are neighbours of the Voronoi diagram.
8. Reorder the Voronoi diagram data structure, overwriting the entries of the deleted vertices.
9. Repeat steps (2-8) for the next point.