Error Estimation for Finite Element Methods

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Abstract

This lecture is meant as an introductory survey of error estimation techniques for finite elements. We consider a simple model problem and we introduce its finite element formulation. Then, we present different a-priori and a-posteriori analyses and consider different error estimation techniques available in the literature. Numerical tests are provided to support the presented theoretical results. For the interested reader, a bibliographic note has been added.

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1 Introduction

Many of the real-world phenomena can be mathematically modelled by means of differential equations, usually expressing conservation laws or other well-known principles of the physics. Exact solutions are unlikely to be found for these equations. This issue has led to the employment of approximate solutions as, for example, those provided by the discretization methods proper of the numerical analysis (finite differences, finite elements, finite volume, etc.). The central role gained nowadays by the computer simulations is certainly a direct consequence of such methodologies.
A reliable control of the error associated with mathematical modelling is, consequently, one of the main goal of the numerical analysis.
Different kinds of error can be responsible for the bad quality of the numerical solution. First, the data provided as input to the adopted numerical scheme may be affected with accidental measurement errors. Then, very often, the chosen differential equations only approximatively describe the analyzed phenomenon. Finally, the approximation procedure itself contributes to the total error.
Our interest is pointed here to the last type of error (usually called *approximation error*) and at how we can measure it in some suitable way.
Approximation error estimation can have two major roles. The first is to provide a level of confidence on the computations. In this case, we may look for means of obtaining, possibly in a cheap way, strict upper and lower bounds to the error associated with a given computation.
Another application of error estimation is linked to mesh adaption, i.e. how to build a computational mesh best suited for the problem at hand. When we have that objective in mind, the capability of the error estimate to mimic correctly the asymptotic behaviour of the error with respect to the mesh size, possibly at local level, becomes of a greater relevance. The error analysis reported here is mostly related to the latter approach.
The present work is conceived as a simple introduction to the subject. We have made an effort to present some basic material which may provide the reader with an understanding of the major techniques for error estimation and mesh adaption, without any pretention of being exhaustive. We will consider a very simple model problem to be able to treat the matter in a rather rigorous way without dwelling on the technicalities required when tackling more complex problems. Furthermore, we will only consider two-dimensional problems and triangular elements. Yet many of the considerations here made may be extended to more general and three-dimensional problems.
For the interested reader we provide a rather extensive bibliography survey which covers more complex problems.

1.1 Some Nomenclature and Main Definitions

For the sake of clarity, we recall here the major nomenclature used in this work.
We will indicate with Ω a domain of $\mathbb{R}^2$ with regular boundary $\partial\Omega$, while $\mathbf{x} = (x, y)^T$ will denote a point of Ω in cartesian coordinates.
We will often use a *multi-index* notation to indicate derivatives. Let $f : \Omega \to \mathbb{R}$ be a real valued function defined on Ω and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ a couple of non-negative integer numbers. We define $|\boldsymbol{\alpha}|$ as the quantity $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$. The derivatives of f will then be indicated as

$$D^{\boldsymbol{\alpha}} f(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(\mathbf{x}).$$