Anisotropic Error Estimation for Finite Element Methods

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Abstract

In these notes we present error estimates which enable to control not only the mesh element size h_K but also its aspect ratio and orientation. These estimates are called anisotropic. They may help in reducing the degrees of freedom required for a given solution accuracy. Here we provide some results for a-priori anisotropic interpolation error estimates, by presenting both standard results of the literature and some recent developments.

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1 Introduction

Solutions exhibiting *lower-dimensional features* (that is, great variations along certain directions with less significant changes along the other ones) are recurrent in many physical phenomena. For instance, shock layers in compressible Navier-Stokes equations or boundary and internal layers in advection-diffusion problems.

Finite element approximations based on *anisotropic meshes*, whose elements are elongated to "follow" the geometry of the solution, may then be advantageous. This fact has been already exploited mainly in compressible computational fluid dynamics, where practical computations are normally carried out on anisotropic meshes ([5, 12, 16, 19, 22]). With an anisotropic mesh the number of degrees of freedom required for a *given solution accuracy* may be considerably reduced compared with standard isotropic techniques.

An anisotropic error estimate should be capable to furnish information on how the error varies, not just with respect to a single mesh spacing parameter, such as the element diameter h_K, but also with respect to the element aspect ratio and orientation. Yet, the derivation of anisotropic estimates requires some additional care compared with the corresponding standard result. We will here consider mainly interpolation error estimates, which we have seen to be one of the building blocks for discretisation error estimates [13].

Anisotropic estimates may be sought in the form (see, e.g., [1])

$$|v - \pi_h^1(v)|_{H^1(K)} \leq C \sum_{\substack{\alpha_1+\alpha_2=1 \\ \alpha_1,\alpha_2\geq 0}} h_{1,K}^{\alpha_1} h_{2,K}^{\alpha_2} \left| \frac{\partial v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|_{H^1(K)}, \tag{1}$$

π_h^1 being the Lagrangian linear finite element interpolant, already introduced in the companion lecture notes [13], while $h_{1,K}$ and $h_{2,K}$ are element length scales associated to the x and y directions, respectively.

This result is in contrast with the corresponding isotropic one (see again [13])

$$|v - \pi_h^1(v)|_{H^1(K)} \leq C h_K \, |v|_{H^2(K)}, \tag{2}$$

where the element diameter h_K obviously satisfies $h_K \geq \max(h_{1,K}, h_{2,K})$. A mesh adaption procedure based on (2) allows only to find an optimal distribution of element diameters, while, using (1), it is in principle possible to control both $h_{1,K}$ and $h_{2,K}$. Therefore, a grid adapted by (1) may have elements with a strong aspect ratio. The main drawback of an estimate in the form (1) is that it demands conditions on the coordinate system and on the maximal allowable mesh angle [1]. These requirements may limit its applicability in automatic mesh adaption procedures.

In these notes, we derive different anisotropic error estimates, by exploiting the transformation from a reference element, typical of standard finite element procedures. The approach is similar to that adopted in [8, 9, 10, 25].

More precisely, after having assigned the basic concepts of transformation from a reference element and of mesh metric function, we give some details on a first interpolation error estimate which relies on the cited results of E.F. D'Azevedo and R.B. Simpson [8, 9]. The estimate is only asymptotically exact and is valid for regular functions. Moreover, it does not provide an upper bound for the error, but just an approximation. Yet, estimates of this type have been extensively used, in particular for compressible flows computations [5, 15, 16, 19]. We will show an example.

In the second part of these notes, we report some recent results on the derivation of more general anisotropic interpolation error estimates and we give an indication on how to use them in practical mesh adaption techniques.