An Introduction to Mesh Quality

M.Berzins
CPDEs Unit, School of Computer Studies, University of Leeds
Woodhouse Lane, Leeds LS2 9JT, U.K.

Abstract

The issue of mesh quality for unstructured triangular and tetrahedral meshes is considered. The theoretical background to finite element methods is used to understand the basis of present-day geometrical mesh quality indicators. Simple tetrahedral mesh quality measures using both geometrical and solution information are described. Some of the issues in mesh quality for unstructured tetrahedral meshes are illustrated by means of simple examples. These examples illustrate that the use of geometrical mesh quality measures alone may give misleading information and that mesh quality for problems modelling complex flows depends on the numerical error, the norm used to measure the error and the relationship between these quantities and the shape of the elements.

1 Introduction

The range of partial differential equations problems (p.d.e.s) solved by finite element and finite volume solvers based on triangular and tetrahedral meshes e.g. [11] [48] is rapidly increasing. The original applications problem class for many such solvers was in the area of solid mechanics and elasticity in particular. These same types of methods are now being applied to a wide range of problems in solid and fluid mechanics ranging from linear elasticity to turbulent flows. This very broad spectrum of applications naturally raises the issue of whether or not the meshes being used are appropriate for the applications being considered. The relationship between the shape of finite elements in unstructured meshes and the error that results in the numerical solution is of increasing importance as finite elements are used to solve problems with highly anisotropic and, often, very complex solutions. Strong local variations in solution component values make it difficult to assess the quality of the mesh for each component without somehow incorporating solution behaviour. In the case of mesh generation, the usual approach is to assume that the solution to the problem is such that mesh quality may be viewed as being independent of the solution [48, 32]. Indeed when no solution has been computed on the mesh this is the only way to proceed. Once a solution has been computed, the generally accepted point of view is that it is both the shape of the elements and the local solution behaviour that are important, particularly for highly directional flow problems [41, 43]. The starting point for this work is the analysis of Babuska and Aziz [10], who showed that the requirement for triangles is that there should be no large angles. This work was extended to tetrahedral elements by Krizek [27] in a similar spirit.

The intention here is not to deal with the issue of how to construct an optimal mesh but instead to consider the related issue of how the quality of an existing mesh should be assessed given a solution. This reflects an important practical issue, particularly in three space dimensions, when a mesh generator produces a mesh of unknown quality for a complex solution. The requirement is then to assess how appropriate the mesh is for the computed solution. The ideal approach is to use a computed error estimate to assess whether or not the mesh should be refined. This error estimate should reflect not only the interpolation error caused by approximating the solution by a finite element space on a given mesh but also the discretization error of the numerical method used to approximate the p.d.e. and the choice of norm used to measure the error.

In many cases however such error estimates are not available but it is still desirable to understand whether or not the mesh is appropriate. This paper will discuss the simple mesh quality indicator of Berzins [16] based on L^2interpolation error estimates. The fundamental assumption being made is that the solution is being represented by a piecewise linear triangular or tetrahedral basis and that the function being approximated is quadratic. This assumption allows the error to be approximated by a quadratic function and the results of Nadler [36, 37] to be used for the triangular case. The resulting indicator has been shown to be related to those of Bank [12] and Weatherill [48] when geometry alone is taken into account.

The quantities used in defining the full indicator have also been used to generate [20] and modify [2] meshes in two dimensions. This paper will show that the new indicator may be used to identify which triangular or tetrahedral elements need refining and also which edge(s) should be refined. A model of boundary layer flow will be used to demonstrate how the indicator performs in identifying which triangle is best. A further simple example will show the optimum mesh will depend critically on the choice of norm used to measure the error.

The second part of the paper will consider the indicator in the case of a linear element tetrahedral mesh. A parameterised tetrahedron combined with a simple model of a solution with highly directional gradients will be used to illustrate how the new indicator identifies the effect of directionality on the linear element approximation error and how this contrasts with a purely geometrical mesh quality measure.

The conclusion of the paper is that while purely geometrical mesh quality indicators may do a good job in identifying meshing anomalies, the appropriateness of a mesh for a given solution cannot be decided using geometry alone and that although indicators based on interpolation errors may give better insight into how to stretch the mesh the only real solution is to use error estimators with an explicit geometry dependence. The material covered in these notes is taken from the surveys of Berzins [18, 19] and is partly described in its original form in [16, 17].

2 Finite Element Interpolation Error Estimates

In order to start to understand the issue of mesh quality it is necessary to review the important finite element results that form a basis for existing mesh quality measures. In order to state these results it is necessary to introduce some notation. Without loss of generality the case of linear finite elements on triangular meshes will be considered. Define the error as being the difference between the linear approximation, u_{lin} and the true solution u i.e. $e_{lin}(x,y) = u(x,y) - u_{lin}(x,y)$. The L^2 error norm over a triangle T is defined by $||e_{lin}(x,y)||_{L^2(T)}$ where

$$||e_{lin}(x,y)||^2_{L^2(T)} = \int_T (e_{lin}(x,y))^2 dxdy \ . \tag{1}$$