Anisotropic Delaunay based mesh generation

P.L. George and H. Borouchaki
INRIA, Domaine de Voluceau, BP 105, Le Chesnay Cedex, France

Abstract

The construction of governed triangulations (and thus governed meshes), either isotropic anisotropic (non-isotropic) is a relatively recent field of interest motivated by different requests. Up to now, two-dimensional anisotropy was produced by mesh modifications (diagonal swap) or specific methods (separate construction of boundary layers before being merged in a mesh constructed by an automatic method). Three-dimensional anisotropy, moreover, is more delicate to obtain (apart using specific ways). This course discuss a systematic manner to produce such meshes by means of a Delaunay style method.

Note : The following material about metric and triangulation is mainly extracted from [4], Chapter 4. Other materials are original. Also a good understanding of [5] and [6] is assumed.

Introduction

The construction of governed triangulations (and thus governed meshes), either isotropic (when the control concerns the size requirement at any space location[1]) or anisotropic (non-isotropic) (when the control concerns both the desired size and directionality) is a relatively recent field of interest motivated by different requests. Among the possible applications are finite element computations for problems where the expected solutions vary very rapidly and/or have directional features. To give some keys about the anisotropic context, one may consider flow problems with shocks or boundary layers, in CFD. Another type of problems, in solid mechanics, provides interesting applications for isotropic triangulations with size constraints or for general anisotropic triangulations. Parametric surface meshing can also be developed by means of anisotropic mesh generation algorithms applied in the parametric space. In a more general context, adaptation problems naturally lead to the construction of triangulations (or more precisely of meshes) subjected to size specification (*i.e.,* with an isotropic control) or to size and direction prescriptions (anisotropic control). In fact, this control is a way to govern the mesh construction as part of an iterative adaptive process so as to conform to the physical features of the given problem.

[1]A specific application example of a mesh subjected to a size criterion is provided by bathymetric meshes. Such meshes are used in water simulations with two-dimensional elements whose size is governed by a third dimension, namely the depth. Apart from this precise example, the construction of a mesh adapted to a metric map, as it is in adaptive problems, is the natural application of a governed mesh generation technique.

Up to now, such triangulations were constructed almost manually using the expertise of the engineers or following some heuristics. Thus, the construction was more or less heuristic, based on local transformations or optimization procedures (see, for instance, [9], [8] and [10] for anisotropic construction schemes).

In this course, we aim to extending the Delaunay incremental method, as described in [4], Chapter 2 and in [6], so as to propose a systematic approach for isotropic size conforming or anisotropic triangulation construction.

Let us consider the space R^d with $d = 2$ or $d = 3$. We assume that a set or a cloud of points (or sites), denoted as $\mathcal{S}$, is provided, in which the points are properly located. We would like to define the environment suitable for the desired extension. The fundamental issue, as will be seen, consists in defining a *metric* everywhere in the space, this metric indicating both the directions to be followed and the expected sizes with respect to these directions. Thus, the aim of a governed isotropic or anisotropic triangulation algorithm is to construct elements conforming to a given size map or to both the given directions and the related sizes. Regarding these anisotropic constraints, the optimal element, in two dimensions, is no longer a classical Euclidean equilateral triangle.

This course will first review some key notions related to the concept of a metric. Then, it will consider the classical triangulation case (*i.e.,* the Delaunay kernel as described in [5]) interpreted in terms of a metric. Finally, the classical case will be extended to the present context (in particular regarding the classical Delaunay kernel). We will show indeed that *the only requirement* is to replace the Euclidean structure implicitly used in the classical situation by a Riemannian structure. Nevertheless, from a computational point of view, it will be seen that these materials are not directly usable thus requiring us to develop discrete approaches so as to propose an effective Delaunay-type algorithm suitable for anisotropic triangulation construction.

1 Notion of a metric

As described in various references, the Delaunay triangulation method relies on a simple construction, referred to as the *Delaunay kernel*, involving the so-called *Delaunay measure*, [4], [5]. This construction is based on a proximity criterion and thus relies on distance comparisons. For this reason, we will now focus on the notion of distance.

1.1 Metrics and distances

Let Ω (R^d or $Conv(\mathcal{S})$ or $Box(\mathcal{S})$ be the considered domain[2] and let X be an arbitrary point in Ω.

Let us assume that a metric or a metric tensor is specified at any point X in Ω, consisting in practice of a $d \times d$ symmetric positive definite matrix $\mathcal{M}(X)$. In two dimensions, such a matrix is defined by :

$$\mathcal{M}(X) = \begin{pmatrix} a_X & b_X \\ b_X & c_X \end{pmatrix}$$

with $a_X > 0, c_X > 0$ and $a_X c_X - b_X^2 > 0$.

The field $(\mathcal{M}(X))_{X \in \Omega}$ is assumed to be continuous. A Riemannian structure is thus obtained in Ω. This field together with this structure is denoted by $(\Omega, (\mathcal{M}(X))_{X \in \Omega})$.

[2] $Conv(\mathcal{S})$ in the convex hull of $\mathcal{S}$. $Box(\mathcal{S})$ is a box, thus a convex set, enclosing all the points in $\mathcal{S}$.