High Performance Computing with Spectral Element Methods

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Abstract

The two lectures are devoted to the spectral element method (SEM) applied to the Navier–Stokes equations for viscous incompressible Newtonian fluids. The first presentation describes the state-of-the-art algorithms for direct numerical simulations. The space discretization relies on the $\mathbb{P}_N - \mathbb{P}_{N-2}$ approximation while the time integrators use an implicit/explicit approach. Solvers and preconditioners are based on the conjugate gradient method. The pressure computation is handled by the Uzawa algorithm which may be preconditioned in several ways. The case of complex geometries leads to the use of domain decomposition methods and various preconditioning techniques. The second lecture will focus on recent developments, namely the arbitrary Lagrangian-Eulerian formulation and the method of moving grid. Two examples are considered: free surface flows and the interaction between a viscous fluid and a Hookean elastic solid. As a final application, large-eddy simulation is analyzed and new models are proposed. The lecture ends with practical remarks on the numerical implmenetation and parallelization of spectral element codes.

Keywords: spectral element, ALE formulation, LES, ADM

AMS Subject Classification: 65M70

1 SEM State-of-the-Art

1.1 The continuous Navier–Stokes equations

The governing equations describing the flow of a viscous Newtonian incompressible fluid of density ρ and dynamic viscosity μ are given by the Navier–Stokes equations

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}\, p + \mu\, \Delta \boldsymbol{v} + \rho \boldsymbol{f} \tag{1}$$

with the incompressibility constraint coming from the mass conservation

$$\operatorname{div} \boldsymbol{v} = 0 \tag{2}$$

In Eq. (1), the symbol D/Dt denotes the material time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla, \tag{3}$$

where $\boldsymbol{v}$ is the velocity field in the Eulerian representation and p the pressure. The body force is denoted by $\boldsymbol{f}$.

1.1.1 Boundary Conditions

The continuum mechanics hypothesis implies that the viscous fluid sticks to the wall. Therefore, one imposes the no-slip wall condition

$$\boldsymbol{v} = \boldsymbol{v}_w, \tag{4}$$

$\boldsymbol{v}_w$ being the prescribed wall velocity.

An interesting class of fluid flow problems is concerned with free surface flows, where a viscous fluid is directly in contact with an essentially inviscid fluid such as air. Some industrial examples are coating flows, extrusion, and crystal growth. These problems are highly nonlinear because the shape of the free surface, where special conditions are to be applied, is also part of the solution itself. Let us introduce the Cauchy principle where the density of contact forces represented by the stress vector $\boldsymbol{t}$ depends on the unit normal $\boldsymbol{n}$ to the surface $\partial\Omega(t)$ at position $\boldsymbol{x}$. More precisely, there exists a tensor field such that

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}, \tag{5}$$

where $\boldsymbol{\sigma}$ denotes the symmetric stress tensor. Since the free surface is in mechanical equilibrium, we have, from Eq. (5),

$$\boldsymbol{t}_{\text{fluid}} + \boldsymbol{t}_{\text{gas}} = \boldsymbol{0}. \tag{6}$$

The projection of this relation onto the normal ($\boldsymbol{n}$) with respect to the viscous fluid and tangent ($\boldsymbol{\tau}$) unit vectors of the surface produces the free surface conditions ($\boldsymbol{n}_{gas} = -\boldsymbol{n}$)

$$\boldsymbol{t}_{\text{fluid}} \cdot \boldsymbol{n} = \boldsymbol{t}_{\text{gas}} \cdot \boldsymbol{n} = -p_{\text{gas}}, \tag{7}$$

$$\boldsymbol{t}_{\text{fluid}} \cdot \boldsymbol{\tau} = 0. \tag{8}$$

Here we have discarded the surface tension effects.

1.1.2 Initial Conditions

The initial condition requires that the velocity field satisfies

$$\boldsymbol{v}(\boldsymbol{x}, t = 0) = \boldsymbol{v}^0(\boldsymbol{x}), \tag{9}$$

with

$$\operatorname{div} \boldsymbol{v}^0 = 0, \tag{10}$$

and the initial boundary conditions. The superscript indicates the time level.

1.2 Weak Formulation: the Galerkin Method

The weak formulation of the Navier–Stokes equations is given in the functional context of Sobolev spaces and the spatial approximation is considered in the framework of the Galerkin method. The weak formulation is built up using appropriate test functions. Integration by parts with the help of a Green identity reduces the order of the partial differential operators and enlarges the space of admissible solutions. In the framework of spectral element methods (SEM) the computational domain is broken up into E elements and the space approximation is performed with high-order Lagrange-Legendre polynomials of degree N that are constructed as local tensor-product nodal bases for multidimensional problems. The rate of convergence may be analyzed for smooth