LECTURE I - A REVIEW OF FLUID MECHANICS

One makes a model of a fluid by asserting that the fluid occupies regions of three dimensional Euclidean space. In the Eulerian description of the fluid's motion, one assumes the existence of functions $\rho(\underline{x},t)$ and $q(\underline{x},t)$ defined on the fluid region, such that

$$\iiint_R \rho(\underline{x},t)\, d\underline{x} = M_R \tag{1}$$

gives the mass of fluid contained in region R at time t, and that, if $\gamma(\underline{x},t)$ is the concentration of any quantity per unit mass of fluid,

$$\iint_S \rho(\underline{x},t)\, \gamma(\underline{x},t)\, \underline{q}(\underline{x},t)\cdot\underline{n}\, dS = \Gamma \tag{2}$$

where $\underline{n}$ is a unit normal to S, gives the flux of that quantity across S in the sense of $\underline{n}$. The functions ρ and $\underline{q}$ are called respectively the density and velocity of the fluid. They must be integrable functions of $\underline{x}$, and we will assume them to be integrable functions of t as well.

A first restriction on ρ and $\underline{q}$ is given by the conservation of mass. Let R be any fixed region in E_3, bounded by the closed surface S. Then, the statement that mass is neither created or destroyed in R is,

$$\left[\iiint_R \rho(\underline{x},t)\, d\underline{x}\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \iint_S \rho(\underline{x},t)\, \left(-\underline{n}\cdot\underline{q}(\underline{x},t)\right)\, dS\, dt \tag{3}$$

where $\underline{n}$ is now the outward unit normal on S. If one assumes that ρ is everywhere continuously differentiable with respect to t, equation (3) implies

$$\int_{t_1}^{t_2} \left[\iiint_R \frac{\partial\rho}{\partial t}\, d\underline{x} + \iint_S \rho\ \underline{n}\cdot\underline{q}\ dS\right] dt = 0 ,$$

and thence, if ρ and $\underline{q}$ are continuously differentiable functions of $\underline{x}$, it follows that

$$\int_{t_1}^{t_2} \iiint_R \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q})\right] d\underline{x}\, dt = 0 \qquad (4)$$

for all sufficiently regular regions R and all time intervals (t_1, t_2). Since by our assumptions the integrand of (4) is continuous, this can be true only if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0 \qquad (5)$$

everywhere in the fluid region. This equation of conservation of mass can also be obtained quite generally from kinetic theory, when the density ρ and velocity $\underline{q}$ appear as certain ensemble averages.

In order to write a dynamic equation, one must consider the action of one part of the fluid on another. Consider a region R of the fluid, bounded by the closed surface S with outward unit normal $\underline{n}$. In the most general model with which I'm familiar, that of a Cosserat continuum, one says that there exist fields $\underline{t}(\underline{x}, \underline{n})$ and $\underline{c}(\underline{x}, \underline{n})$, such that the net force and moment exerted on the fluid in R by the rest of the fluid are given by

$$\underline{F} = \iint_S \underline{t}(\underline{x}, \underline{n})\, dS \; ; \qquad \underline{M} = \iint_S \underline{c}(\underline{x}, \underline{n})\, dS \qquad (6)$$

The field $\underline{c}(\underline{x}, \underline{n})$ is usually neglected, and we will also neglect it.

Then, with a glance at the mechanics of sets of mass points, one takes as a fundamental dynamic principle for continuum mechanics that linear momentum is conserved in R :

$$\left[\iiint_R (\rho q_i)\, dv\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \iint_S \left(t_i(\underline{x}, \underline{n}) + (-\rho q_i n_\ell q_\ell)\right) dS\, dt \qquad (7)$$