The Application of Wiener-Hermite Expansions to Models of Turbulence

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In Nonlinear Problems in Random Theory Norbert Wiener proposed a theory of turbulence in which the velocity fields are expanded in Hermite functionals of a Gaussian white noise function. Meecham (1968) and his coworkers recognized the significance of Wiener's ideas and developed a fixed base expansion which they applied to Burgers' model and in approximate form to fluid turbulence. Canavan and Leith (1968) returned to Wiener's original suggestion in developing a simple Lagrangian theory of turbulence based on an advected white noise function. This note applies both the fixed and moving base expansions to finite dimensional models of turbulence in order to point out as simply as possible their essential features, strengths, and weaknesses.

Consider the expansion

$$v_i = v_i^o + v_i^1 + v_i^2 + \ldots \quad , \; i = 1,2,3,..,N<\infty \tag{1}$$

$$= K_i^o + K_{ij}^1 n_j + K_{ijk}^2 \, (n_j n_k - \delta_{jk}) + \ldots$$

where the K^n's are nonrandom, δ_{jk} is the Kronecker tensor, and repeated subscripts are summed. This is an expansion of the "velocity" $\underline{v}$ in statistically orthogonal multidimensional Hermite polynomials of the Gaussian white noise vector $\underline{n}$ whose components are real Gaussian random variables with zero means and statistics defined by

$$\langle n_j n_k \rangle = \delta_{jk} \tag{2}$$

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The Gaussianity of $\underline{n}$ dictates that cumulant statistics such as

$$<n_j n_k n_\ell n_m> - <n_j n_k> <n_\ell n_m> - <n_j n_\ell> <n_k n_m> \tag{3}$$

$$- <n_j n_m> <n_k n_\ell> = 0$$

vanish, so that higher moments of the noise base are readily evaluated. Truncation of the expansion in Eq. (1) to two terms allows $\underline{v}$ to be the most general Gaussian random vector, since

$$<v_i> = v_i^o \tag{4}$$

$$<v_i v_j> = K^1_{ir} K^1_{js} <n_r n_s> + K^o_i K^o_j = K^1_{ir} K^1_{jr} + K^o_i K^o_j \tag{5}$$

and all higher moments of the two term $\underline{v}$ are determined by these. Use equations (2) and (3) to evaluate

$$<v^2_i v^1_j v^1_k> = K^2_{i\ell m} K^1_{jp} K^1_{kq} <(n_\ell n_m - \delta_{\ell m}) n_p n_q> = 2K^2_{ipq} K^1_{jp} K^1_{kq} \tag{6}$$

The evaluation of this moment illustrates three important features of Wiener-Hermite expansions. First, truncation of the expansion at $\underline{v}^2$ allows for nonvanishing triple correlations without which the statistics of fluid turbulence cannot be explained. Second, the randomness of the velocity is due to the white noise base; the nonrandom kernels, K^j, in the expansion determine the statistics of $\underline{v}$. Third, all statistics of $\underline{v}$ are realizable. Realizability of a moment simply means that it can be