Some Properties of a Lagrangian Wiener-Hermite Expansion*

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ABSTRACT

Wiener proposed that the turbulent velocity be expanded in Hermite functionals of a Gaussian white noise random function advected by the fluid. This paper describes the mechanics of converting his suggestion into a computable model, and assesses its range of validity as an approximation for incompressible, homogeneous, and isotropic turbulence. The terms retained are a linear term, $\tilde{v}^1$, representing an arbitrary Gaussian velocity, and a quadratic term, $\tilde{v}^2$, representing a non-Gaussian contribution to the velocity needed for energy transfer. The requirement that advection by the dependent velocity $\tilde{v} = \tilde{v}^1 + \tilde{v}^2$ not alter the statistics of the base necessitates a further truncation of the base to antisymmetric quadratic basis elements. Realizability of any statistics of $\tilde{v}$ is common to all Wiener-Hermite expansions. The projected equations for the Lagrangian expansion conserve energy by nonlinear interaction, preserve the inviscid Gaussian equipartition ensemble, and are invariant to random Galilean transformations. Numerical calculations with an approximate form of these equations reveal that irreversible relaxation to the inviscid equipartition solution is not a property of the Lagrangian model, and that the rapid convergence advanced as the original motivation for studying Wiener-Hermite expansions does not survive closure by truncation.

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The dynamics of the model is not inconsistent with the existence of an inertial range. A simple numerical search routine failed to produce a solution corresponding to such an equilibrium ensemble.

Wiener (1958) proposed a simple Lagrangian theory of turbulence in which the turbulent velocity fields are expanded in Hermite functionals of a Gaussian white noise random function advected by the fluid. The significance of Wiener's ideas was recognized in a series of papers by Siegel and Meecham (1959), Meecham and Siegel (1959, 1964), Imamura, Meecham, and Siegel (1965), Siegel, Imamura, and Meecham (1965), and Meecham and Jeng (1968) in which was developed a fixed base expansion applicable to fluid turbulence. They and their students, Su (1967) and Kahng (1968), applied the theory to Burgers' model and in approximate form to fluid turbulence. Despite the optimistic tone of Saffman's (1968) readable introduction to this version of the theory, it has serious flaws which have been pointed out by Orszag and Bissonnette (1967) and Crow and Canavan (1967).

Some of the flaws which render the stationary base expansion invalid as an approximation to high Reynolds number turbulence are overcome by the expansion in an advected base which Wiener originally proposed. This paper presents the equations for Wiener's Lagrangian expansion, examines their analytical and numerical predictions, and assesses their range of validity as an approximation for incompressible, homogeneous, and isotropic turbulence. The model guarantees certain realizability and consistency properties which have eluded some other straightforward theories. It offers a novel solution to the closure problem encountered in treating nonlinear problems statistically, but one which leads to difficulties in numerical calculations.

A number of mathematical problems are avoided by using a wave-vector space representation and treating quantities defined on the discrete array of wave vectors

$$\underset{\sim}{k} = \frac{2\pi}{L} \underset{\sim}{m} \; ; \; m_i = 0, \pm 1, \pm 2, \ldots \tag{1}$$

corresponding to flows cyclic in cubes of side L. In this representation Gaussian white noise is a complex random vector field so defined (Canavan and Leith 1968) as to have the two point joint statistics

$$\begin{aligned} \langle n_i (\underset{\sim}{k}) \, n_j (\underset{\sim}{k}') \rangle &= \delta_{ij} \, \delta(\underset{\sim}{k} + \underset{\sim}{k}') \\ \delta(\underset{\sim}{0}) = 1 \; ; \; \delta(\underset{\sim}{k}) &= 0, \; \underset{\sim}{k} \neq \underset{\sim}{0} \end{aligned} \tag{2}$$

in terms of which all higher moments of the noise can be written. The linear and quadratic Hermite polynomials

$$\begin{aligned} h_i^1 (\underset{\sim}{k}, t) &= n_i(\underset{\sim}{k}, t) \\ h_{ij}^2 (\underset{\sim}{k}, \underset{\sim}{m}, t) &= n_i (\underset{\sim}{k}, t) \, n_j (\underset{\sim}{m}, t) - \delta_{ij} \delta(\underset{\sim}{k} + \underset{\sim}{m}) \end{aligned} \tag{3}$$

can then be used to write the expansion of the velocity